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Analytically based estimation of the maximum amplitude during passage through resonance

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Abstract

Approximation formulae for both the maximum amplitude and the respective resonance frequencies of a linear oscillator passing resonance are derived analytically. Starting from the exact solution for run-up or run-down with constant acceleration, irrelevant terms are omitted. The analytically found results are compared to the known empirical approximation formulae.

Keywords: Analytical; Estimation; Non-stationary; Resonance; Amplitude

1. Introduction

During run-up or run-down, machines are subject to oscillating forces of increasing or decreasing frequency. If resonances have to be passed, the maximum amplitudes are a major criterion for the design of the machine.

In the case of non-stationary processes there is, in general, an interaction between the oscillating system and its driving mechanism (Kononenko, 1969; Christ, 1966; Wauer, 1976; Markert et al., 1977, 1980; Gasch, 1979). While approaching a resonance, the driving mechanism pumps energy into the oscillation, whereas after the passage of the resonance, a part of the vibrational energy flows back into the driving mechanism. This leads to a lengthening of the resonance passage during run-up and to a shortening during run-down (Markert, 1980). In the extreme, it may lead to stalling in the resonance during run-up. However, if the driving mechanism is strong, the reaction of the oscillator to its driving mechanism may be neglected; then, the excitation of the oscillating system is independent of its vibrational state and the excitation frequency is prescribed by the driving mechanism.

While the resonance frequencies and the stationary oscillations of linear, time-invariant systems can be easily calculated, the analytical calculation of the time or frequency response for non-stationary operation is rather complicated and in practice rarely performed.

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Nevertheless, the maximum amplitude during the passage of resonance remains an important design criterion for many machines. Therefore, several authors empirically developed approximate formulae for the maximum resonance amplitudes of linear oscillators excited by non-stationary forces or unbalances based on numerical integration data (Lewis, 1932; Katz, 1947; Zeller 1949; Fearn and Millsaps, 1967; Markert, 1988; Irretier and Leul, 1993). An overview is given by Leul (1994). Although some of these approximation formulae yield good approximations for the non-stationary resonance amplitude, they have the restriction of not being mathematically proven. In this article, a different approach is presented. Starting from the exact solution for run-up or run-down through resonance with constant angular acceleration, approximate formulae for both the maximum amplitude and the respective resonance frequency are derived analytically.

2. Equation of motion

The general equation of motion

$$m\ddot{x}(t) + b\dot{x}(t) + cx(t) = a_0 p(t) + a_1 \dot{p}(t) + a_2 \ddot{p}(t) + a_3 \dddot{p}(t) \tag{1}$$

describes the forced motion of one-dimensional, time-invariant, linear oscillators for various types of excitation mechanisms. The time-dependent state variable $x(t)$ is governed by the system parameters, mass m, damping b and stiffness c on the one hand, and by the type of excitation on the other hand. Mostly, the excitation is a linear combination of the excitation function $p(t)$ and its derivatives in time, with constant coefficients a_0, a_1, a_2 and a_3. The state variable $x(t)$ and the excitation function $p(t)$ can have various physical meanings, e.g., absolute or relative displacements, velocities, angles or forces. For illustration, the simple system shown in Fig. 1 shall be considered (Witfeld, 1990).

A homogeneous cylinder with mass m_w and radius r rolls on a cart which performs the prescribed motion $y(t)$. The cylinder is connected to the cart by a spring c and a damper b.

The equation of motion for the angular displacement of the cylinder

$$\tfrac{3}{2} m_w \ddot{\varphi}(t) + b\dot{\varphi}(t) + c\varphi(t) = -\frac{m_w}{r}\ddot{y}(t) \tag{2}$$

shows the well-known inertia term as excitation and the agreement with the general equation (1) yields $x(t) = \varphi(t)$ and $p(t) = y(t)$ with parameters $m = 3m_w/2$, $a_0 = a_1 = a_3 = 0$ and $a_2 = -m_w/r$.

If alternatively, the absolute vibration $x_S(t)$ of the cylinder's center of gravity is chosen as the state variable, the equation of motion results in

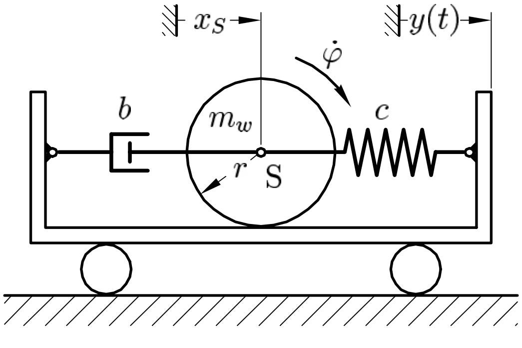

Fig. 1. Oscillating cylinder on a moving cart.

$$\tfrac{3}{2}m_w\ddot{x}_S(t) + b\dot{x}_S(t) + cx_S(t) = cy(t) + b\dot{y}(t) + \tfrac{1}{2}m_w\ddot{y}(t). \tag{3}$$

In this case, on the right-hand side, not only the second, but also the zeroth and the first time derivative of the excitation function $y(t)$ occur. Now the correspondence is $x(t) = x_S(t)$ and $p(t) = y(t)$ with the excitation parameters $a_0 = c$, $a_1 = b$, $a_2 = m_w/2$ and $a_3 = 0$.

Finally, if one introduces the sum $F(t)$ of the spring and the damper force – the so-called foundation force – as the state variable, then the equation of motion

$$\tfrac{3}{2}m_w\ddot{F} + b\dot{F} + cF = -cm_w\ddot{y} - bm_w\dddot{y} \tag{4}$$

contains also the third time derivative of the excitation function $y(t)$ and the corresponding parameters are $a_0 = a_1 = 0$, $a_2 = -cm_w$ and $a_3 = -bm_w$.

It should be mentioned that dealing with aperiodic excitation functions, in particular with non-stationary run-up and run-down processes, it is in most cases favorable not to carry out the differentiation of the excitation function explicitly, but to eliminate the differentiation during the calculations by partial integration. In this way, unnecessary assumptions or even mistakes can be avoided that are sometimes found in the literature, e.g., the assumption that during non-stationary operation the unbalance excitation remains proportional to the square of the varying excitation frequency.

For further considerations, it is useful to make the excitation function $p(t)$, the state variable $x(t)$ and the parameters dimensionless by means of the undamped natural frequency

$$\omega_0 = \sqrt{\frac{c}{m}} \tag{5}$$

and judiciously chosen reference quantities $\hat{x}$ and $\hat{p}$. With the dimensionless time

$$\tau = \omega_0 t \tag{6}$$

and the resulting derivative rule

$$\frac{\mathrm{d}\ldots}{\mathrm{d}t} = \omega_0\frac{\mathrm{d}\ldots}{\mathrm{d}\tau} = \omega_0(\ldots)^{\circ}, \tag{7}$$

as well as the non-dimensional excitation function $P(\tau) = p(t)/\hat{p}$ and state variable $X(\tau) = x(t)/\hat{x}$, the general equation of motion under consideration

$$\overset{\circ\circ}{X}(\tau) + 2D\overset{\circ}{X}(\tau) + X(\tau) = A_0P(\tau) + A_1\overset{\circ}{P}(\tau) + A_2\overset{\circ\circ}{P}(\tau) + A_3\overset{\circ\circ\circ}{P}(\tau) \tag{8}$$

contains only a few system inherent parameters, which are the damping ratio

$$D = \frac{b}{2\sqrt{cm}} \tag{9}$$

and the excitation coefficients

$$A_0 = \frac{\hat{p}}{c\hat{x}}a_0, \quad A_1 = \omega_0\frac{\hat{p}}{c\hat{x}}a_1, \quad A_2 = \omega_0^2\frac{\hat{p}}{c\hat{x}}a_2, \quad A_3 = \omega_0^3\frac{\hat{p}}{c\hat{x}}a_3. \tag{10}$$

3. Eigenvalues and harmonic excitation

In both stationary and non-stationary operations, the behavior of the system is primarily characterized by the two eigenvalues

$$\lambda_1 = -D + \mathrm{i}\sqrt{1-D^2} \;\text{ and }\; \lambda_2 = -D - \mathrm{i}\sqrt{1-D^2}, \tag{11}$$

which are complex conjugate for damping coefficients $|D| < 1$.

If the excitation is harmonic in time with the excitation frequency $\Omega = \eta\omega_0$,

$$P(\tau) = \cos(\eta\tau + \beta), \tag{12}$$

after transient vibrations have decayed, the remaining oscillations are of the form:

$$X_p(\tau) = V(\eta)\cos[\eta\tau + \beta - \delta(\eta)]. \tag{13}$$

This oscillation is characterized by the amplification factor:

$$V(\eta) = \sqrt{\frac{(A_0 - \eta^2 A_2)^2 + \eta^2 (A_1 - \eta^2 A_3)^2}{(1-\eta^2)^2 + (2D\eta)^2}}, \tag{14}$$

and the phase angle which is determined by

$$\sin\delta(\eta) = \frac{2D\eta(A_0 - \eta^2 A_2) - \eta(A_1 - \eta^2 A_3)(1-\eta^2)}{V(\eta)[(1-\eta^2)^2 + (2D\eta)^2]}, \tag{15}$$

$$\cos\delta(\eta) = \frac{(A_0 - \eta^2 A_2)(1-\eta^2) + 2D\eta^2(A_1 - \eta^2 A_3)}{V(\eta)[(1-\eta^2)^2 + (2D\eta)^2]}. \tag{16}$$

These real functions are combined in the system's complex transfer function:

$$H(\eta) = V(\eta)\mathrm{e}^{-\mathrm{i}\delta(\eta)} = \frac{(A_0 - \eta^2 A_2) + \mathrm{i}\eta(A_1 - \eta^2 A_3)}{(1-\eta^2) + \mathrm{i}2D\eta}, \tag{17}$$

which depends on the excitation frequency η, but not on the time τ.

4. Run-up or run-down with constant acceleration

Exact solutions are known for run-up or run-down processes with constant angular acceleration. Older examinations of Pöschl (1933), Weidenhammer (1958), as well as Goloskokow and Filippow (1971) on oscillators excited by force or unbalance were the basis for a general study by Markert and Pfützner (1981) which encompasses all possible excitation mechanisms simultaneously.

During run-up or run-down with constant angular acceleration $\mathring{\mathring{\varphi}} = \alpha$, the excitation has constant amplitude,

$$P(\tau) = \cos\varphi(\tau). \tag{18}$$

Starting from the initial values η_0 and β, the angular frequency of the excitation,

$$\mathring{\varphi}(\tau) = \frac{\Omega(t)}{\omega_0} = \alpha\tau + \eta_0, \tag{19}$$

increases or decreases linearly and the phase angle,

$$\varphi(\tau) = \tfrac{\alpha}{2}\tau^2 + \eta_0\tau + \beta, \tag{20}$$

changes quadratically with time τ.

The non-stationary system response $X(\tau)$ can be packed into the form

$$X(\tau) = |Q(\tau)|\cos[\varphi(\tau) - \psi(\tau)] \tag{21}$$

comparable to Eq. (13) resulting from stationary excitation. But contrary to the transfer function $H(\tau)$ for stationary excitation, the complex amplitude function,

$$Q(\tau) = |Q(\tau)| e^{-i\psi(\tau)}, \tag{22}$$

is time-dependent in the non-stationary case. Otherwise, $|Q(\tau)|$ and $\psi(\tau)$ describe the non-stationary oscillations analogously as V and δ describe the stationary oscillations.

The complex amplitude function (for $D \neq 1$),

$$Q(\tau) = B_1 w(v_1) + B_2 w(v_2) + \left\{E_1 \overset{\circ}{\varphi}(\tau) + E_0\right\} + \left\{C_1 e^{v_{10}^2 - v_1^2} + C_2 e^{v_{20}^2 - v_2^2}\right\}, \tag{23}$$

is described by the so-called error function (Abramowitz and Stegun, 1970)

$$w(u) = e^{-u^2}\left(1 + \frac{2i}{\sqrt{\pi}} \int_0^u e^{u^{*2}} du^*\right) \tag{24}$$

and depends on the two complex times:

$$v_1(\tau) = -\frac{1+i}{2\sqrt{\alpha}}\{\alpha\tau + \eta_0 + i\lambda_1\}, \quad v_{10} = v_1(0), \tag{25}$$

$$v_2(\tau) = \operatorname{sign}\alpha \frac{1+i}{2\sqrt{\alpha}}\{\alpha\tau + \eta_0 + i\lambda_2\}, \quad v_{20} = v_2(0). \tag{26}$$

The constants

$$\begin{aligned}
B_1 &= \frac{1-i}{4\sqrt{1-D^2}} \sqrt{\frac{\pi}{\alpha}} (A_0 + \lambda_1 A_1 + \lambda_1^2 A_2 + \lambda_1^3 A_3), \\
B_2 &= \frac{\operatorname{sign}\alpha(1-i)}{4\sqrt{1-D^2}} \sqrt{\frac{\pi}{\alpha}} (A_0 + \lambda_2 A_1 + \lambda_2^2 A_2 + \lambda_2^3 A_3), \\
E_1 &= iA_3, \\
E_0 &= A_2 - 2DA_3, \\
C_1 &= \frac{(\lambda_2 X_0 - \overset{\circ}{X}_0)e^{-i\beta}}{\lambda_2 - \lambda_1} - B_1 w(v_{10}) + \frac{A_1 + (i\eta + \lambda_1)A_2 + (i\alpha - \eta^2 + i\eta\lambda_1 + \lambda_1^2)A_3}{\lambda_2 - \lambda_1}, \\
C_2 &= \frac{(\lambda_1 X_0 - \overset{\circ}{X}_0)e^{-i\beta}}{\lambda_1 - \lambda_2} - B_2 w(v_{20}) + \frac{A_1 + (i\eta + \lambda_2)A_2 + (i\alpha - \eta^2 + i\eta\lambda_2 + \lambda_2^2)A_3}{\lambda_1 - \lambda_2}
\end{aligned} \tag{27}$$

contain the system parameters on the one hand and the initial conditions $X_0 = X(0)$ and $\overset{\circ}{X}_0 = \overset{\circ}{X}(0)$ on the other hand.

As an example, a run-down of the oscillator described by Eq. (2), excited by an unbalance, is shown in Fig. 2. A high initial frequency of excitation $\eta_0 = 10$ was chosen, so that possible transient oscillations would have faded before the image border at $\overset{\circ}{\varphi} = 2$ is reached. The plot clearly shows the difference of the envelope function $|Q(\tau)|$ of the non-stationary oscillations and the stationary resonance curve $V(\eta)$, as well as the three main characteristics of the non-stationary passage of a resonance:

(1) In non-stationary operations, the maximum amplitude $|Q|_{max}$ is always smaller than in stationary resonance operations. The faster the resonance is passed, the smaller is the non-stationary resonance amplitude.

(2) The maximum amplitude $|Q|_{max}$ of the oscillations does not occur at the moment when the exciting frequency $\dot{\varphi}(t)$ and the natural frequency ω_0 coincide, but later at the frequency η_R. Thus, the maximum amplitude shifts towards higher frequencies during run-up and towards lower frequencies during run-down.

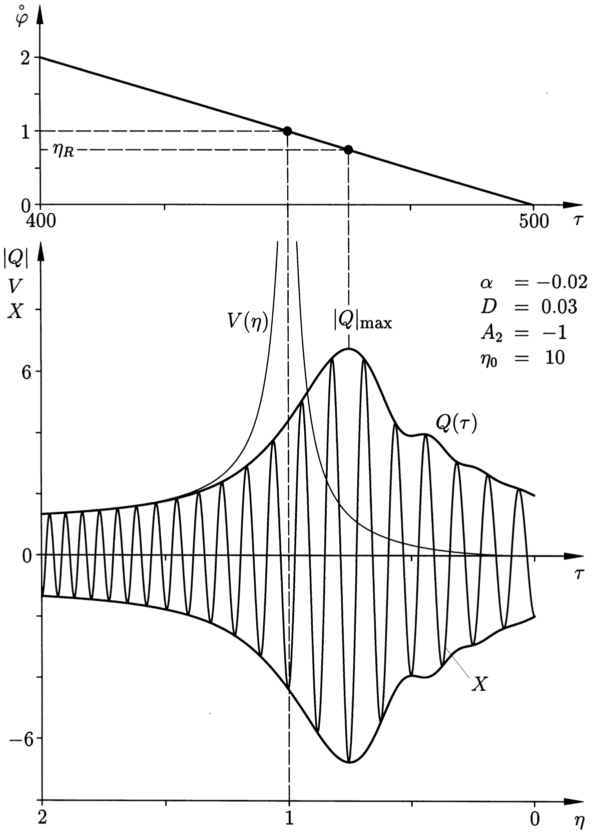

Fig. 2. Run-down of an oscillator excited by unbalance.

(3) After passing the resonance, a low-frequency oscillation of the amplitude occurs, which can be interpreted as the superposition of the two parts of the oscillation: free oscillations with the natural frequency, initiated during the passage of the resonance, and forced oscillations with variable excitation frequency.

The exact equation (23) can also be used approximately for systems with slowly varying natural frequency $\omega(t)$. For this purpose, instead of the actual angular acceleration α, an effective acceleration α_{eff} is used, calculated as the difference between the actual acceleration and the rate of change of the natural frequency (Leul, 1994):

$$\alpha_{eff} = \alpha - \frac{1}{\omega_0^2} \frac{d\omega(t)}{dt}. \qquad (28)$$

The analytical solution (23) applies to any damping ratios $D \neq 1$. For the critical case of $D = 1$ with double eigenvalue $\lambda_1 = \lambda_2 = -1$, there is also an analytical solution (Markert, 1982) which resembles solution (23) in its structure. However, the question of the maximum amplitude does not arise with such a high damping.

The shown solution method can also be applied to linear multi-degree-of-freedom systems and continua by modal decomposition into a corresponding number of one-degree-of-freedom systems.

5. The error function

In the solution formula (23) for the non-stationary run-up or run-down, the so-called error function (24) occurs with the complex arguments v_1 and v_2. These complex variables change linearly with time, and therefore, they can be interpreted as complex times. Because of the ambiguity of the square root $\sqrt{\alpha}$, there are two possible definitions for each v_1 and v_2. Contrary to former publications (Markert, 1980; Markert and Pfützner, 1981), the sign convention was chosen with regard to the later amplitude estimation in such a way that only one term dominates in the solution. In Fig. 3, the range of the complex times corresponding to this definition is sketched. These move through the complex plane along 45° lines. At the moment of the passage of the resonance, the imaginary part of $v_1(\tau)$ is negative, while $v_2(\tau)$ has always a positive imaginary part if $\overset{\circ}{\varphi} \geqslant 0$.

Analyzing the contour plot of the error function (Fig. 4), it becomes apparent that $|w(v_2)|$ is always much smaller than $|w(v_1)|$ and is therefore negligible for an estimation of the non-stationary resonance amplitude.

The error function satisfies the differential equation:

$$w'(v) = -2vw(v) + \frac{2i}{\sqrt{\pi}}. \tag{29}$$

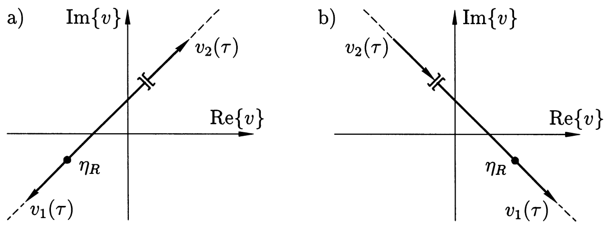

Fig. 3. Paths of the complex variables $v_1(\tau)$ and $v_2(\tau)$: (a) during run-up and (b) during run-down.

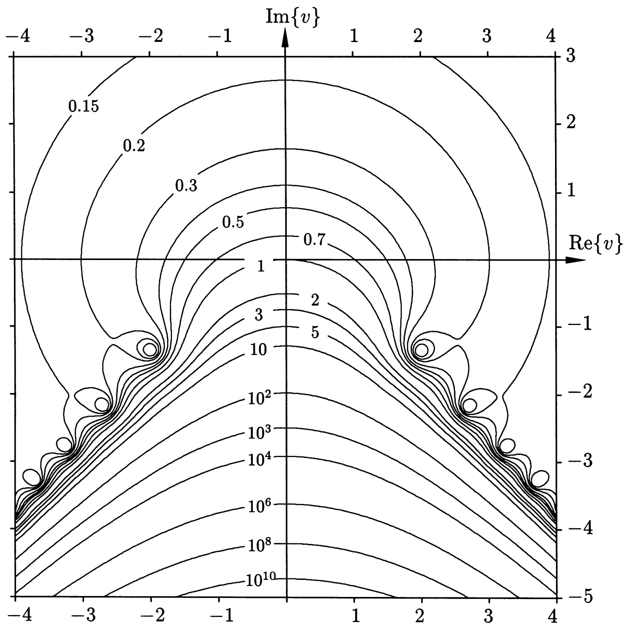

Fig. 4. Contour lines of the error function (Abramowitz and Stegun, 1970).

For numerical calculations, there exist power series expansions, asymptotic expansions and also approximation equations (Markert, 1982). The continued fraction,

$$w(u) = \frac{\mathrm{i}}{\sqrt{\pi}} \left(\frac{1}{u-} \frac{1/2}{u-} \frac{1}{u-} \frac{3/2}{u-} \frac{2}{u-} \cdots \right), \tag{30}$$

which is valid for arguments u with positive imaginary parts, $\mathrm{Im}\{u\} > 0$, is particularly useful for the numerical evaluation. The three relations,

$$w(-\bar{u}) = \bar{w}(u), \tag{31}$$

$$w(-u) = 2\mathrm{e}^{-u^2} - w(u), \tag{32}$$

$$w(\bar{u}) = 2\mathrm{e}^{-\bar{u}^2} - \bar{w}(u), \tag{33}$$

in which bars mark complex conjugate values, allow the transformation of the argument from one quadrant to any other and thereby into the domain where the continued fraction (30) is valid.

6. Approximations for the passage through resonance

According to Eq. (23), the complex, time-dependent amplitude function $Q(\tau)$ consists of four additive parts.

The term

$$S_4(\tau) = C_1 \mathrm{e}^{v_{10}^2 - v_1^2} + C_2 \mathrm{e}^{v_{20}^2 - v_2^2} \tag{34}$$

describes free oscillations which normally have decayed before the resonance zone is reached and thus does not contribute to the resonance amplitude. Therefore, this term can be neglected for approximating the non-stationary oscillations within the resonance area.

The term

$$S_3(\tau) = E_1 \overset{\circ}{\varphi}(\tau) + E_0 = A_2 + A_3[\mathrm{i}\overset{\circ}{\varphi}(\tau) - 2D] \tag{35}$$

only occurs if the equation of motion includes the second or third derivative of the excitation function. In all cases, it contributes only a small part to the resonance amplitude $|Q|_{\max}$. It is constant for $A_3 = 0$ and it changes only slightly in time for $A_3 \neq 0$. Therefore, $S_3(\tau)$ can be replaced by a constant within the resonance area and can be neglected for the calculation of the non-stationary resonance frequency.

The term

$$S_2(\tau) = B_2 w(v_2) \tag{36}$$

comes to resonance if $\mathrm{i}\overset{\circ}{\varphi}(\tau)$ gets close to the second eigenvalue λ_2, which implies that the absolute value of $\mathrm{i}\overset{\circ}{\varphi} - \lambda_2 = D + \mathrm{i}(\overset{\circ}{\varphi} + \sqrt{1 - D^2})$ becomes small. For positive excitation frequencies $\overset{\circ}{\varphi} \geqslant 0$, S_2 remains small and can be neglected as well.

The remaining term

$$S_1(\tau) = B_1 w(v_1) \tag{37}$$

is the dominating part of the total solution and characterizes the passage of the resonance as well as the subsequent oscillations of the amplitude. It reaches resonance after the dimensionless excitation frequency $\overset{\circ}{\varphi}(\tau)$ passes the value 1.

Among the four terms (34)–(37), only $S_1(\tau)$ and $S_3(\tau)$ contribute noticeably to the resonance curve $Q(\tau)$, $S_1(\tau)$ being the dominating part. Accordingly, a good approximation for the non-stationary passage of resonances is

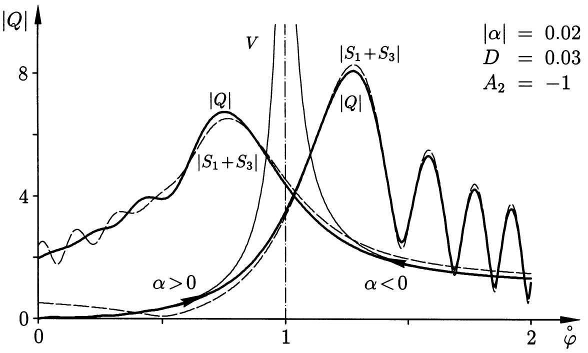

Fig. 5. Absolute values of the complex amplitude $Q(\tau)$ (—) and its approximation $S_1(\tau) + S_3(\tau)$ (– – –) for a run-up ($\alpha > 0$) and a run-down ($\alpha < 0$).

$$Q(\tau) \approx B_1 w(v_1) + \left\{A_2 + A_3[\mathrm{i}\overset{\circ}{\varphi}(\tau) - 2D]\right\}. \tag{38}$$

Fig. 5 illustrates this for the oscillator with unbalance excitation in Eq. (2). In the resonance area, the differences between the exact solutions and their approximations are very small, in the present numerical example less than 4%.

Even if the approximation (38) is already considerably simplified in comparison with the exact solution $Q(\tau)$, it is still inconvenient to evaluate because of the complex-valued error function $w(v_1)$. To evaluate $w(v_1)$, one either has to consult a comprehensive handbook of mathematical tables (Abramowitz and Stegun, 1970), or use a computer program. The computation becomes much easier if the error function $w(v_1)$ is approximated with a continued fraction. To apply the continued fraction according to Eq. (30), the argument v_1 has to be transformed by using the symmetry relations (31)–(33), so that its imaginary part is positive. Depending on the number of terms used in the continued fraction, different approximations for the error function result:

$$w(v_1) \approx w_1(v_1) = 2\mathrm{e}^{-v_1^2} + \frac{\mathrm{i}}{\sqrt{\pi}}\frac{1}{v_1}, \tag{39}$$

$$w(v_1) \approx w_2(v_1) = 2\mathrm{e}^{-v_1^2} + \frac{\mathrm{i}}{\sqrt{\pi}}\frac{2v_1}{2v_1^2 - 1}, \tag{40}$$

$$w(v_1) \approx w_3(v_1) = 2\mathrm{e}^{-v_1^2} + \frac{\mathrm{i}}{\sqrt{\pi}}\frac{2(v_1^2 - 1)}{v_1(2v_1^2 - 3)}. \tag{41}$$

Of course, these approximations only hold if the imaginary part of v_1 is negative, i.e., for run-ups ($\alpha > 0$) if $\overset{\circ}{\varphi} > \sqrt{1 - D^2} + D$ and for run-downs ($\alpha < 0$) if $\overset{\circ}{\varphi} < \sqrt{1 - D^2} - D$, so in all cases after the stationary resonance frequency and therefore always within the region of the non-stationary resonance frequency. The resulting approximations for the complex amplitude function $Q(\tau)$ are of different accuracy,

$$\begin{aligned}
Q_1(\tau) &= B_1\left\{2\mathrm{e}^{-v_1^2} + \frac{\mathrm{i}}{\sqrt{\pi}}\frac{1}{v_1}\right\} + \left\{A_2 + A_3(\mathrm{i}\overset{\circ}{\varphi} - 2D)\right\},\\
Q_2(\tau) &= B_1\left\{2\mathrm{e}^{-v_1^2} + \frac{\mathrm{i}}{\sqrt{\pi}}\frac{2v_1}{2v_1^2 - 1}\right\} + \left\{A_2 + A_3(\mathrm{i}\overset{\circ}{\varphi} - 2D)\right\},\\
Q_3(\tau) &= B_1\left\{2\mathrm{e}^{-v_1^2} + \frac{\mathrm{i}}{\sqrt{\pi}}\frac{2(v_1^2 - 1)}{v_1(2v_1^2 - 3)}\right\} + \left\{A_2 + A_3(\mathrm{i}\overset{\circ}{\varphi} - 2D)\right\}.
\end{aligned} \tag{42}$$

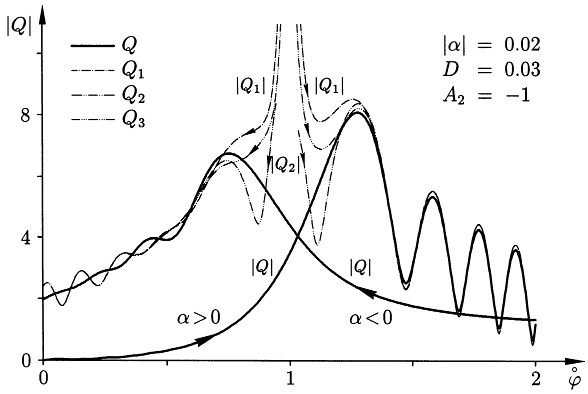

Fig. 6. Absolute values of the complex amplitude function $Q(\tau)$ and its approximations $Q_1(\tau)$, $Q_2(\tau)$ and $Q_3(\tau)$ for a run-up and a run-down.

Fig. 6 compares the exact solution to the different approximations. After the maximum $|Q|_{\max}$ of the amplitude, already the most simple approximation $Q_1(\tau)$, which involves only one term of the continued fraction, comes close to the exact solution. However, estimating the corresponding non-stationary resonance frequency η_R requires the approximation $Q_2(\tau)$ in most cases.

7. Estimation of the non-stationary resonance amplitude and frequency

The preceding investigations showed that the most important characteristics of the non-stationary behavior are formed by the part $S_1(\tau)$ of the complete solution $Q(\tau)$. All other terms are small, and if $S_3(\tau)$ constitutes a part of the amplitude function at all, it does not change significantly with the excitation frequency $\overset{\circ}{\varphi}$. Therefore, the excitation frequency $\overset{\circ}{\varphi} = \eta_R$ at which the maximum amplitude $|Q|_{\max}$ occurs can be determined from the condition:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\{Q(\tau)\bar{Q}(\tau)\} \approx \frac{\mathrm{d}}{\mathrm{d}\tau}\{B_1 w(v_1)\bar{B}_1\bar{w}(v_1)\} = 0. \tag{43}$$

Omitting the real constant $B_1\bar{B}_1$, applying the relations $w(v_1) = \bar{w}(-\bar{v}_1)$ and $w'(-\bar{v}_1) = -\bar{w}'(v_1)$, and considering the identity $z + \bar{z} = 2\mathrm{Re}\{z\}$, one obtains

$$\begin{aligned} T(\tau) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\{w(v_1)\bar{w}(v_1)\} = \frac{\mathrm{d}}{\mathrm{d}\tau}\{w(v_1)w(-\bar{v}_1)\} = w'(v_1)\bar{w}(v_1)\overset{\circ}{v}_1 + \bar{w}'(v_1)w(v_1)\bar{\overset{\circ}{v}}_1 \\ &= 2\mathrm{Re}\{w'(v_1)\bar{w}(v_1)\overset{\circ}{v}_1\} = 0. \end{aligned} \tag{44}$$

Finally, after eliminating the derivative $w'(v_1)$ by Eq. (29), the condition for the non-stationary resonance frequency η_R is given by

$$T(\tau) = 2\mathrm{Re}\left\{\sqrt{\alpha}(1+\mathrm{i})\bar{w}(v_1)\left[\frac{\mathrm{i}}{\sqrt{\pi}} - v_1 w(v_1)\right]\right\} = 0. \tag{45}$$

Fig. 7 shows $T(\tau)$ as a function of the excitation frequency $\overset{\circ}{\varphi}$ in the undamped case $D = 0$. The zeros of the function $T(\tau)$ yield not only the non-stationary resonance frequency η_R corresponding to the amplitude maximum $|Q|_{\max}$, but all other excitation frequencies at which the amplitude function $|Q(\tau)|$ exhibits relative maxima and minima.

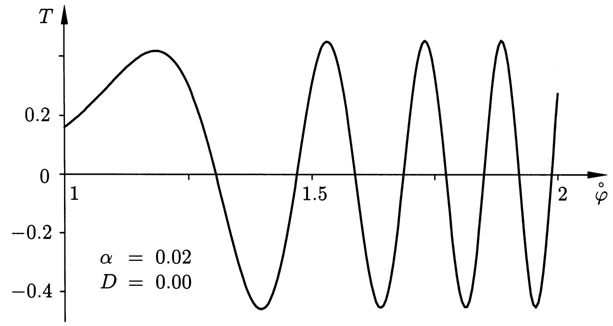

Fig. 7. Determination of the zeros of the function $T(\tau)$.

Condition (45) can be solved numerically for arbitrary values of damping. However, various numerical examples show that the damping D, as long as it remains small, does not influence the location η_R of the non-stationary resonance maximum considerably. Furthermore, $|Q(\tau)|$ is rather flat in the resonance zone even without damping, so that it is not necessary to calculate η_R exactly for estimating $|Q|_{max}$. Therefore, damping is taken into account for estimating the non-stationary resonance amplitude $|Q|_{max}$, whereas the non-stationary resonance frequency η_R is calculated without damping.

In general, the function T depends on the three parameters α, $\overset{\circ}{\varphi}$ and D. For the given values of angular acceleration and damping, the zeros of T characterize the excitation frequencies corresponding to the maxima and minima of the amplitude $|Q|$. In the case of zero damping $D = 0$, the number of independent variables in T can be reduced to one by using the new variable,

$$\delta = \frac{(\overset{\circ}{\varphi} - 1)^2}{2|\alpha|}, \tag{46}$$

which is real and positive, yielding

$$T(\tau) = 2\sqrt{|\alpha|}\mathrm{Re}\left\{ \bar{w}\left(-(1+\mathrm{i})\sqrt{\frac{\delta \,\mathrm{sign}\,\alpha}{2}} \right) \left[(\mathrm{i}-1)\sqrt{\frac{\mathrm{sign}\,\alpha}{\pi}} + 2\mathrm{i}\sqrt{\frac{\delta}{2}} w\left(-(1+\mathrm{i})\sqrt{\frac{\delta \,\mathrm{sign}\,\alpha}{2}} \right) \right] \right\} = 0. \tag{47}$$

Eq. (47) is valid for both run-up ($\alpha > 0$) and run-down ($\alpha < 0$). It has an infinite number of solutions δ_n ($n = 0, 1, 2, \ldots$), from which the corresponding excitation frequencies can be calculated according to

$$\overset{\circ}{\varphi}_n = 1 + \mathrm{sign}\,\alpha \sqrt{2\delta_n |\alpha|}. \tag{48}$$

The numerical evaluation shows that the values of δ are the same for run-up and run-down with equal absolute values of angular acceleration. For both processes, the resonance shift is the same in this approximation. Approximate values for δ_n can be found by using the truncated continued fraction (30),

$$\delta_n \approx \frac{\pi}{4}(4n - 1). \tag{49}$$

Odd values of n belong to the maxima and even values to the minima of $|Q(\tau)|$. In Table 1, the exact solutions of Eq. (47) in comparison with the approximate values according to Eq. (49) are listed.

The solution $\delta_1 = 2.3272$ describes the location of the maximum amplitude, $|Q|_{max}$, which occurs when the excitation frequency $\overset{\circ}{\varphi}$ coincides with the non-stationary resonance frequency,

$$\eta_R = 1 + 2.157\,\mathrm{sign}\,\alpha \sqrt{|\alpha|}. \tag{50}$$

Table 1
Zeroes of Eq. (47)

n	δ_n (exact)	δ_n Eq. (49)
1	2.3272	2.36
2	5.508	5.50
3	8.634	8.64
4	11.78	11.78
5	14.92	14.92

Eq. (50) is an analytically justified approximation formula for the excitation frequency η_R corresponding to the non-stationary resonance maximum $|Q|_{max}$, which provides satisfactory results even for $D \neq 0$. The non-stationary resonance maximum $|Q|_{max}$ can be calculated by substituting η_R into one of the approximation formulae for the amplitude $Q(\tau)$. The numerical computation of the resulting formula is still inconvenient and, therefore, a more simplified relation for the maximum amplitude is determined by further manipulations and approximations.

For small damping, the imaginary part of the eigenvalue λ_1 may be replaced by the value 1:

$$\operatorname{Im}\{\lambda_1\} = \sqrt{1-D^2} \approx 1 \tag{51}$$

yielding

$$v_{1R} \approx -\frac{1+\mathrm{i}}{2\sqrt{\alpha}}\left[\operatorname{sign}\alpha\sqrt{2\delta_1|\alpha|} - \mathrm{i}D\right], \tag{52}$$

$$v_{1R}^2 \approx \mathrm{i}\left(\delta_1 \operatorname{sign}\alpha - \frac{D^2}{2\alpha}\right) + D\sqrt{\frac{2\delta_1}{|\alpha|}}, \tag{53}$$

which simplifies the two terms in Eq. (42) to

$$\frac{1-\mathrm{i}}{4\sqrt{1-D^2}}\sqrt{\frac{\pi}{\alpha}}\frac{\mathrm{i}}{\sqrt{\pi}v_{1R}} \approx -\frac{\operatorname{sign}\alpha\sqrt{2\delta_1|\alpha|}+\mathrm{i}D}{2(2\delta_1|\alpha|+D^2)}, \tag{54}$$

$$\frac{1-\mathrm{i}}{4\sqrt{1-D^2}}\sqrt{\frac{\pi}{\alpha}}2\mathrm{e}^{-v_{1R}^2} \approx \sqrt{\frac{\pi}{2\alpha}}\mathrm{e}^{-D\sqrt{\frac{2\delta_1}{|\alpha|}}-\mathrm{i}\left(\frac{\pi}{4}+\delta_1\operatorname{sign}\alpha-\frac{D^2}{2\alpha}\right)}. \tag{55}$$

Considering the approximation

$$\mathrm{e}^{-\mathrm{i}\left(\frac{\pi}{4}+\delta_1\operatorname{sign}\alpha\right)} \approx \frac{-1}{\sqrt{\operatorname{sign}\alpha}}, \tag{56}$$

the final approximation for the non-stationary resonance amplitude is

$$|Q|_{max} \approx \left|\left(A_2+\mathrm{i}\left(1+\operatorname{sign}\alpha\sqrt{2\delta_1|\alpha|}\right)A_3\right) - \operatorname{sign}\alpha(A_0+\mathrm{i}A_1-A_2-\mathrm{i}A_3)\left[\sqrt{\frac{\pi}{2|\alpha|}}\mathrm{e}^{\left(\mathrm{i}\frac{D^2}{2\alpha}-D\sqrt{\frac{2\delta_1}{|\alpha|}}\right)} + \frac{\sqrt{2\delta_1|\alpha|}+\mathrm{i}D\operatorname{sign}\alpha}{2(2\delta_1|\alpha|+D^2)}\right]\right|. \tag{57}$$

This approximation is valid for arbitrary angular accelerations and damping ratios D. It also provides correct results for the limiting cases $\alpha \to 0$ or $D \to 0$. For example, for pure force excitation ($A_0 = 1$), the limit $\alpha \to 0$ leads to the well-known approximation formula:

$$|Q|_{\max} \approx \frac{1}{2D}, \tag{58}$$

and the limit $D \to 0$ yields the good result:

$$|Q|_{\max} \approx \frac{1}{0.6732\sqrt{|\alpha|}}. \tag{59}$$

If the angular acceleration is not very small, $\alpha \gg 5D^2$, Eq. (57) can be further simplified by developing the absolute value of the exponential function into a power series, neglecting the higher-order terms and the small imaginary parts in the last two expressions:

$$|Q|_{\max} \approx \Bigg| \left(A_2 + \mathrm{i}\left(1 + \operatorname{sign}\alpha\sqrt{2\delta_1|\alpha|}\right)A_3\right) - \operatorname{sign}\alpha(A_0 + \mathrm{i}A_1 - A_2 - \mathrm{i}A_3) \times \left[\sqrt{\frac{\pi}{2|\alpha|}}\left(1 - D\sqrt{\frac{2\delta_1}{|\alpha|}} + D^2\frac{\delta_1}{|\alpha|}\right) + \frac{1}{2\sqrt{2\delta_1|\alpha|}}\right]\Bigg|. \tag{60}$$

Being valid for arbitrary excitation mechanisms, this formula simplifies immensely for a specific mechanism of excitation.

For example, in the special case of force excitation ($A_0 = 1$), one obtains

$$|Q|_{\max} \approx \sqrt{\frac{\pi}{2|\alpha|}}\left(1 - D\sqrt{\frac{2\delta_1}{|\alpha|}} + D^2\frac{\delta_1}{|\alpha|}\right) + \frac{1}{2\sqrt{2\delta_1|\alpha|}} \tag{61}$$

and by reordering the terms finally,

$$|Q|_{\max} \approx \frac{1}{0.673\sqrt{|\alpha|}} - 2.70\frac{D}{|\alpha|} + 2.92\frac{D^2}{\sqrt{|\alpha|^3}}. \tag{62}$$

For the special case of excitation by the damper ($A_1 = 1$), the same approximation results as for force excitation.

If the system is excited by the movement of the foundation ($A_0 = 1$ and $A_1 = 2D$), which is relevant for vibration absorbers, one obtains just the additional factor $\sqrt{1 + 4D^2}$.

In the discussed simplest approximation, these three special cases have the same maximum amplitudes during run-up and run-down.

However, for unbalance excitation ($A_2 = -1$), Eq. (60) yields

$$|Q|_{\max} \approx \sqrt{\frac{\pi}{2|\alpha|}}\left(1 - D\sqrt{\frac{2\delta_1}{|\alpha|}} + D^2\frac{\delta_1}{|\alpha|}\right) + \frac{1}{2\sqrt{2\delta_1|\alpha|}} + \operatorname{sign}\alpha, \tag{63}$$

which contains the additional term $\operatorname{sign}\alpha$, describing the difference in the maximum amplitudes during run-up and run-down by the value 2.

8. Comparison to empirical formulae

In the literature, several approximate formulae can be found for the maximum amplitude $|Q|_{\max}$ and the corresponding excitation frequency η_R during non-stationary passage through resonances. The base of most of these formulae are extensive numerical examinations and subsequent parameter fit for a given equation structure. In the following, an overview of these formulae is given. For comparison, the formulae from the literature have been rearranged and converted into the same notation as used in this article.

Usually, the *approximations for the non-stationary resonance frequency* from the literature do not distinguish between force and unbalance excitation since remarkable differences only arise for large values of the angular acceleration $|\alpha|$. The formula for the non-stationary resonance frequency η_R by Fearn and Millsaps (1967),

$$\eta_R = 1 + 2.15\,\mathrm{sign}\,\alpha\sqrt{|\alpha|}, \tag{64}$$

has the same structure as the analytically based formula (50). Katz (1947) provides the approximation formula

$$\eta_R = 1 + \mathrm{sign}\,\alpha\frac{2.178\sqrt{|\alpha|}}{1 + 0.56D/\sqrt{|\alpha|} + 0.0784D^2/|\alpha|}, \tag{65}$$

which also takes damping into account. As mentioned above, the influence of damping is very weak: even for a damping ratio of 5%, the non-stationary resonance frequency η_R differs less than 4% from the undamped case. In a former publication of Markert (1988), the formula

$$\eta_R = \sqrt{1 - D^2} + \mathrm{sign}\,\alpha\sqrt{2(\pi - 1)|\alpha| + D^2} \tag{66}$$

was given, which in the undamped case ($D = 0$) yields

$$\eta_R = 1 + 2.07\,\mathrm{sign}\,\alpha\sqrt{|\alpha|}. \tag{67}$$

Leul (1994) gave the formula

$$\eta_R = 1 + \mathrm{sign}\,\alpha\frac{2.222\sqrt{|\alpha|}}{1 + 0.556D/\sqrt{|\alpha|}}. \tag{68}$$

The analytically based approximation (50),

$$\eta_R = 1 + 2.157\,\mathrm{sign}\,\alpha\sqrt{|\alpha|}, \tag{69}$$

confirms the empirical estimations and shows that the deviation of the non-stationary resonance frequency from the stationary one is approximately proportional to $\sqrt{|\alpha|}$. The formulae differ from each other slightly in the numerical factors; the values vary from 2.06 to 2.22, while the analytical approximation has the factor 2.157 (Table 2). The effect of these differences is small, as can be seen in Fig. 8.

In the literature, force and unbalance excitation are distinguished from each other for the estimation of the *non-stationary resonance amplitude* $|Q|_{\max}$.

For *force excited systems*, Lewis (1932) gives

$$|Q|_{\max} = \frac{1}{0.68\sqrt{|\alpha|}} - 0.353\,\mathrm{sign}\,\alpha, \tag{70}$$

Fearn and Millsaps (1967),

$$|Q|_{\max} = \frac{1}{0.68\sqrt{|\alpha|}} - 0.25\,\mathrm{sign}\,\alpha + 0.025\sqrt{|\alpha|}, \tag{71}$$

Irretier and Leul (1993),

$$|Q|_{\max} = \frac{1}{\left(0.76\sqrt{|\alpha|} + 2D - 0.51\sqrt{D}\sqrt[4]{|\alpha|}\right) - \mathrm{sign}\,\alpha\left(0.01\sqrt{|\alpha|} + 0.07\sqrt{D}\sqrt[4]{|\alpha|}\right)}, \tag{72}$$

Table 2
The factors of the term $\sqrt{\alpha}$ in different approximation formulae

Author	Factor in $\lvert Q\rvert_{\max}$ force excitation	Factor in $\lvert Q\rvert_{\max}$ unbalance excitation	Factor in η_R
Analytical	0.6732	0.6732	2.157
Fearn and Millsaps (1967)	0.68		2.15
Katz (1947)			2.178
Markert (1988)	0.71		2.07
Irretier and Leul (1993)	0.75/0.77	0.75/0.77	2.222
Lewis (1932)	0.68		
Zeller (1949)	0.80		
Dorning (1959)		0.68/0.71	
Fernlund (1963)		0.71	
Hirano et al. (1968)		0.66	

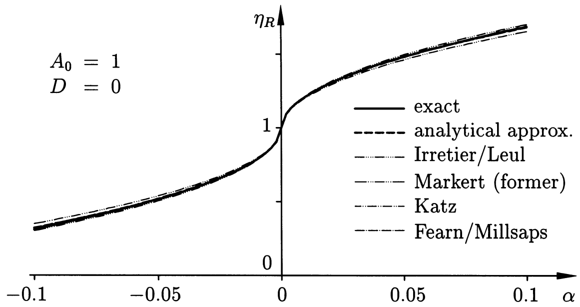

Fig. 8. Non-stationary resonance frequency η_R.

Markert (1988),

$$|Q|_{\max} = \frac{1}{2D + \sqrt{|\alpha|/2}} \approx \frac{1}{0.71\sqrt{|\alpha|} + 2D}, \tag{73}$$

and Zeller (1949),

$$|Q|_{\max} = \frac{1}{2D\sqrt{1-D^2}}\left[1 - e^{-D\sqrt{2\pi/|\alpha|}}\right] \approx \frac{1}{0.80\sqrt{|\alpha|}} - \frac{D\pi}{2|\alpha|}. \tag{74}$$

The corresponding analytically developed approximation formula reduces to

$$|Q|_{\max} \approx \frac{1}{0.673\sqrt{|\alpha|}} - 2.70\frac{D}{|\alpha|} + 2.92\frac{D^2}{\sqrt{|\alpha|^3}}. \tag{75}$$

Fig. 9 contrasts the different approximations to the numerically calculated exact curve.

For *unbalance excited oscillators*, the empirically found approximation formulae for the non-stationary resonance maximum are given by Dorning (1959) as,

$$|Q|_{\max} = \frac{1}{(0.694 - 0.016\,\mathrm{sign}\,\alpha)\sqrt{|\alpha|}}, \tag{76}$$

Fernlund (1963),

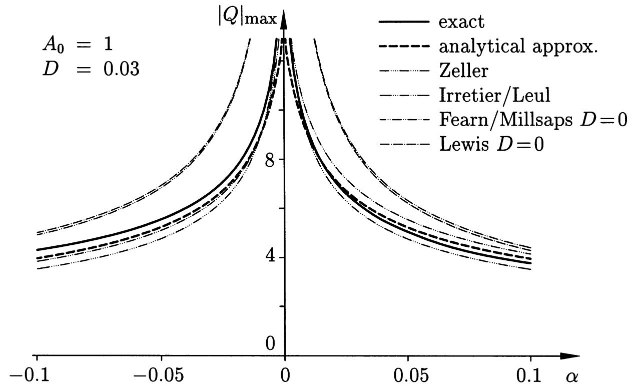

Fig. 9. Maximum amplitude $|Q|_{max}$ of an oscillator with force excitation.

$$|Q|_{max} = \frac{1}{0.71\sqrt{|\alpha|}}, \tag{77}$$

Hirano et al. (1968),

$$|Q|_{max} = \frac{1}{0.66\sqrt{|\alpha|}}\left[e^{-0.9396|\alpha|^{-0.379}D^{0.7}}\right], \tag{78}$$

and Leul (1994),

$$|Q|_{max} = \frac{1 + 0.165\sqrt{|\alpha|} + 0.815\,\mathrm{sign}\,\alpha\sqrt{|\alpha|}}{\left(0.76\sqrt{|\alpha|} + 2D - 0.51\sqrt{D}\sqrt[4]{|\alpha|}\right) - \mathrm{sign}\,\alpha\left(0.01\sqrt{|\alpha|} + 0.07\sqrt{D}\sqrt[4]{|\alpha|}\right)}. \tag{79}$$

The corresponding analytically based approximation is

$$|Q|_{max} \approx \frac{1}{0.673\sqrt{|\alpha|}} - 2.70\frac{D}{|\alpha|} + 2.92\frac{D^2}{\sqrt{|\alpha|^3}} + \mathrm{sign}\,\alpha. \tag{80}$$

Fig. 10 contrasts the different approximations to the numerically calculated exact curve.

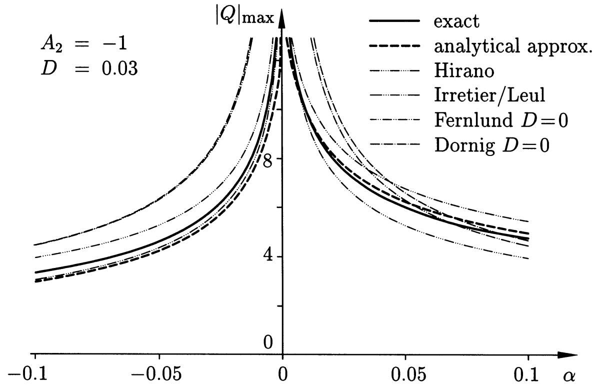

Fig. 10. Maximum amplitude $|Q|_{max}$ with unbalance excitation.

Even if the individual approximations look quite different at first sight, they nevertheless have an essential characteristic in common. For weak damping, the non-stationary resonance amplitude $|Q|_{max}$ is inversely proportional to $\sqrt{|\alpha|}$. The factors of $\sqrt{|\alpha|}$ are in the range between 0.66 and 0.80 and the deviations from the exact value are mostly smaller than 10% (Table 2). The different approximation formulae differ from each other noticeably only in the higher-order terms, so that differences become apparent only for extreme parameter values.

9. Conclusion

The analytically based formulae for the non-stationary resonance amplitude and frequency encompass many possible excitation mechanisms. The formulae have a simple structure and are easy to evaluate. The deviations from the exact values are mostly smaller than 10%.

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